Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Collaborative Techniques in Modal Analysis

M.L. Amirouche*
University of Illinois at Chicago
Chicago, Illinois
and
R.L. Huston†
University of Cincinnati, Cincinnati, Ohio

Introduction

THIS Note presents a new hybrid procedure for determining vibration characteristics of a large structure. The procedure combines modal analysis techniques with recently developed techniques of finite-segment modeling. Kane's equations are used to obtain the governing equations of motion. The procedure is applicable with structures having either linear or nonlinear flexibility and damping characteristics.

It was shown by Huston[1] that flexibility and compliance effects can be modeled by springs and dampers between two adjoining bodies. Specifically, it is shown in Ref. 1 that the governing equations of motion can be written so that bending moments, torsional moments, shearing forces and extension forces at the system joints occur with one component per equation. This representation of the equations of motion provides for the approximation of stiffness and damping by using data from experimental modal and analysis.

Many large structural systems (for example, space stations and cranes) present significant control problems due to the system size. The basic difficulty arises in modeling highly flexible structures. Such structures are known to have a large number of packed modes at very low frequencies.[9-11] The procedure developed herein forms a basis for determining these frequencies and the corresponding mode shapes. The procedures are ideally suited for "tree-like" structures.

The Note itself is divided into five parts. The first part provides background and preliminary concepts. Kinematics and dynamics are discussed in the second and third parts, while part four describes the incorporation of experimental modal analysis, and the final part discusses applications.

Preliminary Concepts

Consider a tree structure as shown in Fig. 1. The geometry of this system can be organized by knowing two arrays: the lower body array $L_0(k)$ and the connecting body array $L_1(k)$. The numbering scheme is desribed in Ref. 3. The topology of the tree structure is represented by a tree array called Λ_k^j. This tree array is used in the description of the kinematics needed for the analysis of the system's dynamics.

Coordinates Systems Shifter Transformation Matrices

Let each body have six degrees-of-freedom: three for rotation and three for translation. Let these degrees-of-freedom for the N bodies of the system be characterized by the coordinates $X_\ell(\ell=1,..., 6N)$. Further let these coordinates be divided into $2N$ sets of triplets describing the translation and rotation of the respective bodies. Consider two typical adjacent bodies B_j and B_k as shown in Fig. 2. The transformation matrix between these two bodies may be defined as

$$SJK_{pq} = \boldsymbol{n}_{jp} \cdot \boldsymbol{n}_{kq} \qquad (p,q=1,2,3) \tag{1}$$

where $\boldsymbol{n}_{ij}$ are mutually perpendicular unit vectors fixed in B_j and B_k. The evaluation of the transformation matrix between any two connecting bodies is performed in accordance with the connection configuration of the tree-array. For example, if the connecting body is nine and the lower body of nine is five, then the transformation matrix between each body from five to nine is the identity matrix (assuming all bodies are aligned in a particular branch). Otherwise the matrix evaluation is dependent upon the orientation of two bodies.

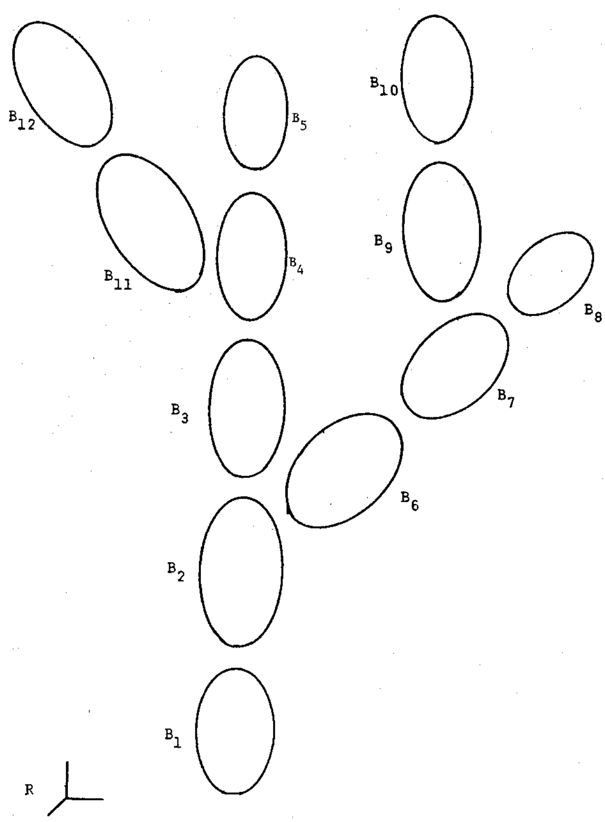

Fig. 1 Tree-like structure.

Submitted Jan. 25, 1985; revision received April 1, 1985, presented as Paper 85-0632 at the AIAA 26th Structures, Structural Dynamics and Materials Conference, Orlando, Fla., April 15-17, 1985.

*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

†Professor of Mechanics, Department of Mechanical & Industrial Engineering. Fellow AIAA.

Table 1 $\omega_{k\ell m}$ partial angular velocities of Fig. 1

ℓk	1,2,3	4,5,6	7,8,9	10,11,12	13,14,15	16,17,18	19,20,21	22,23,24	25,26,27	28,29,30	31,32,33	34,35,36
1	I	0	0	0	0	0	0	0	0	0	0	0
2	I	I	0	0	0	0	0	0	0	0	0	0
3	I	I	I	0	0	0	0	0	0	0	0	0
4	I	I	I	I	0	0	0	0	0	0	0	0
5	I	I	I	I	I	0	0	0	0	0	0	0
6	I	I	0	0	0	2-6 S	0	0	0	0	0	0
7	I	I	0	0	0	2-6 S	I	0	0	0	0	0
8	I	I	0	0	0	2-6 S	I	I	0	0	0	0
9	I	I	0	0	0	2-6 S	I	0	7-9 S	0	0	0
10	I	I	0	0	0	2-6 S	I	0	7-9 S	I	0	0
11	I	I	I	I	0	0	0	0	0	0	4-11 S	0
12	I	I	I	I	0	0	0	0	0	0	4-11 S	I

Kinematics

Angular Velocity

References 1 and 4 show that the angular velocity of a typical body K with respect to an inertial reference frame R can be written as

$$ {}^{R}\omega^{B_k} = \omega_{k\ell m}\dot{X}_\ell n_{om} $$

$$ K=1,\ldots,N \quad \ell=1,\; 6N \quad m=1,2,3 \tag{2} $$

where $\omega_{k\ell m}$ is known as the "partial angular velocity array." The coefficients $\omega_{k\ell m}$ are functions of the relative orientation of adjacent bodies.

Angular Acceleration

By differentiation in Eq. (2), the angular acceleration of body K with respect to R *is*

$$ {}^{R}\alpha^{B_k} = (\dot{\omega}_{k\ell m}\dot{X}_\ell + \omega_{k\ell m}\ddot{X}_\ell)n_{om} \tag{3} $$

for "small" oscillation $\dot{\omega}_{k\ell m}\dot{X}_\ell$ is small compared to $\omega_{k\ell m}\ddot{X}_\ell$. Hence for small oscillation, Eq. (3) may be reduced to

$$ {}^{R}\alpha^{B_k} = \omega_{k\ell m}\ddot{X}_\ell n_{om} \quad (k=1,N \quad \ell=1,6n \quad m=1,2,3) \tag{4} $$

As with the angular velocities it is seen that the angular accelerations are functions of the "Partial angular velocity arrays." Table 1 presents a listing of these arrays.

Mass Center Velocities

References 1 and 3 also show that the velocity and acceleration of the mass centers of the bodies of the system R may be written as

$$ V_k = V_{k\ell m}\dot{X}_\ell n_{om} \tag{5} $$

and

$$ a_k = (\dot{V}_{k\ell m}\dot{X}_\ell + V_{k\ell m}\ddot{X}_\ell)n_{om} \tag{6} $$

where for "small" oscillations $\dot{V}_{k\ell m}\dot{X}_\ell$ is small compared to $V_{k\ell m}\ddot{X}_\ell$. Hence for small oscillation a_k reduces to

$$ a_k = V_{k\ell m}\ddot{X}_\ell n_{om} \tag{7} $$

Table 2 Properties of aluminum test rod

Mass	0.086 kg	0.0196 slug
Length	0.89 m	35. in.
Diameter	0.13 m	0.5 in.
Elastic modulus	6.89×10^{10} N/m	10×10^{6} lb/in.

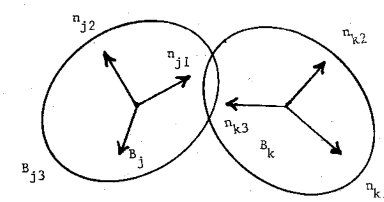

Fig. 2 Two-typical adjoining bodies.

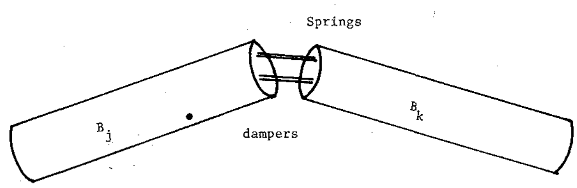

Fig. 3 Two-typical bodies with springs and dampers.

There is a sum from 1 to $6N$ on ℓ and from 1 to 3 on m, k goes from 1 to N (no sum). $V_{k\ell m}$ is known as "the partial velocity array" and is defined as follows: For $\ell\leq 3N$

$$ V_{k\ell m} = \omega_{k\ell m} r_{kn}\mathrm{SRK}e_{mst} + \sum_{p=2}^{k} (\omega_{p-1,\ell,m}\mathrm{SR}(P-1)q_{pn}e_{mst}) \tag{8} $$

and for $\ell\leq 3N$

$$ V_{k\ell m} = \omega_{k,\ell-3N,m}\,(3N<\ell\leq 6N) \tag{9} $$

where r_{kn} are the components of r_k, the q_{kn} are the n_{on} components of q_k, and the e_{mst} are the permutation coefficients defined in Ref. 3.

Table 3 Natural frequencies of free-free rod

Method	Analytical (Ref. 12)	Finite-Element (SAP IV)	Experimental (Modal analysis)	Finite segment modeling
ω_1	68.585	71.59	84.375	70.1
ω_2	189.05	197.3	206.25	191.05
ω_3	370.59	386.6	396.875	378.0
ω_4	612.66	638.8	653.125	636.5
ω_5	915.25	954.0		970.0

Equations of Motion

Using Kane's equations[5-7] together with the procedures developed by Huston et al.,[4,5] the equations of motion of a tree-like structure of flexible bodies can be written in the form

$$a_{\ell p}\ddot{X}_\ell = F_\ell \quad (\ell, p = 1, \ldots, 6N) \tag{10}$$

where the $a_{\ell p}$ known as "generalized mass coefficients" are given by

$$a_{\ell p} = m_k V_{k\ell m} + I_{kmn}\omega_{k\ell m}\omega_{kpn} \tag{11}$$

m_k is the mass of body B_k and I_{kmn} are the n_{om} and n_{on} components of the mass center based inertia dyadic of B_k. As before, there is a sum over the range of the repeated indices. The coefficients of $a_{\ell p}$ are functions of the arrays $\omega_{k\ell m}$, and V_{kpm}. F_ℓ is the generalized active force associated with X_ℓ. It has been found, see Ref. 1, that the governing equations can be written so that bending moment, torsional moment, shearing force, and extension force components at the system joints occur singly—that is, one per equation. Furthermore in Ref. 3 it is shown that the generalized forces can be written as

$$F_\ell = -K_{\ell p}X_\ell - C_{\ell p}\dot{X}_\ell + f_\ell^*(t) \tag{12}$$

where

$K_{\ell p}$ = stiffness matrix

$C_{\ell p}$ = damping matrix

$f_\ell^*(t)$ = forcing function

The stiffness and damping matrices are used in modeling the springs and dampers to account for local flexibility and compliance effects at the connecting joints as discussed in Refs. 1, 3, and 8. It is a relatively simple task to write computer algorithm for the numerical evaluation of the coefficients of Eqs. (9).

Consider two adjoining bodies as depicted in Fig. 3. Let the flexibility characteristics of the bodies be modeled by springs and dampers between them. Let each spring and damper have six degrees of freedom, three for rotation and three for translation. The equations of motion of the structure will then have the form

$$a_{\ell p}\ddot{X}_\ell + C_{\ell p}\dot{X}_\ell + K_{\ell p}X_\ell = f_\ell^*(t) \tag{13}$$

Since the forces and moment components occur once per equation, the stiffness and damping matrices could be represented by diagonal matrices. Note that the terms dropped in Eqs. (3) and (6) due to small oscillation *could be added* where non-linear equations of motion are needed to describe large motions without changing the basic formulation of Eq. (13).

Example Application

To validate and to illustrate the procedure, the vibration characteristics of a long, cylindrical aluminum rod were studied. Table 2 contains a listing of the physical and geometrical properties of the rod.

The natural frequencies due to bending of rod with free ends were then determined. Table 3 presents a comparison of the results. The parameters in the stiffness matrix are functions of the elastic modulus, the inertia dyadic, and the rod length. Hence, by using dimensional analysis, a scaling law may be developed for similar larger stuctures. Suppose a similar structure has a length parameter ℓ and a radial parameter r which are related to the model length and radial parameters as

$$\hat{\ell} = C_\ell \ell \qquad \hat{r} = C_r r \tag{14}$$

From finite-element analysis is has been seen[2] that the stiffness coefficients have the form

$$k_{ii} = C\frac{EI}{\ell^n} = \bar{C}\frac{r^4}{\ell^n} \tag{15}$$

where C and $\bar{C}$ are suitable constants. Hence, the corresponding stiffness coefficient k_{ii} for the similar structure is

$$\hat{k}_{ii} = C\frac{\hat{r}^4}{\hat{\ell}^n} = \bar{C}\frac{C_r^4}{C_\ell^n}\frac{r^4}{\ell^n} = \frac{C_r^4}{C_\ell^n}k_{ii} \overset{D}{=} \xi k_{ii} \tag{16}$$

Therefore, the stiffness coefficients determined by modal analysis of the model structure are simply scaled by the factor ξ defined by Eq. (16).

Conclusions

The scaling procedure developed shows that the stiffness, and also the damping coefficients for large structures may be related to analogous coefficients for modal analysis with those of finite-segment modeling. Hence the vibration and dynamical characteristics of large structures may be obtained.

Specifically, the following procedures are suggested:

1) Develop a finite segment model of the large object structure.

2) Obtain governing dynamic equations of motion for the model.

3) Construct a physical model of the large structure.

4) Determine vibration characteristics of the physical model by modal analysis.

5) Validate and adjust the stiffness and damping coefficient of the finite-segment model by using the modal analysis results.

6) Scale the resulting stiffness and damping coefficients to accommodate the dimensions of the large structure.

7) Use the "scaled" finite-seqment model to predict the dynamic response of the large, object structure.

References

[1]Huston, R.L., "Multi-Body Dynamics Including the Effects of Flexibility and Compliance," *Computers and Structures,* Vol. 14, 1981, pp. 443-451.

[2]Huston, R.L. and Passerello, C.E., *Finite Element Methods—An Introduction,* Marcel-Dekker, 1984.

[3]Amirouche, M.L., "Flexibility and Transient Effects in Multi-Body Systems," Ph.D. Thesis, University of Cincinnati, Cincinnati, Ohio, June 1984.

[4]Huston, R.L. and Passerello, C.E., "On Lagrange's Form of d'Alembert's Principle," *The Matrix and Tensor Quarterly*, Vol. 23, 1973, pp. 109-112.

[5]Kane, T.R. and Levinson, D.A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, 1980, pp. 99-112.

[6]Kane, T.R., "Dynamics of Nonholomic Systems," *Journal of Applied Mechanics*, Vol. 28, 1961, pp. 574-578.

[7]Kane, T.R., Likins, P.W., and Levinson, D.A., *Spacecraft Dynamics*, McGraw-Hill, New York, 1983.

[8]Amirouche, M.L. and Huston, R.L., "Flexibility and Transient Effects of a Multi-Arm Robot," *Proceedings of 7th Symposium EAM, Robotics and Computer-Aided Engineering*, Toronto, Canada, June 1984.

[9]Likins, P.W., "Analytical Dynamics and Non-rigid Spacecraft Simulation," NASA TR 32-1593, 1984.

[10]Wang, S.J. and Cameron, J.M., "Dynamics and Control of a Large Space Antenna," *Journal of Guidance, Control and Dynamics*, Vol. 7, Jan. 1984, p. 69.

[11]Amirouche, M.L. and Huston, R.L., "Human Body Vibrations; Part I, *Proceedings of 1st Mid-Central Conference on Ergonomics/Human Factors*, Cincinnati, Ohio, March 1984.

[12]Young, D. and Felgar, R.P., Jr., *Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam*, The University of Texas Publication, July 1949.

Reducing the Effects of Model Reduction due to Parameter Variations

J.M. Lin*
National Chiao-Tung University, Taiwan, China
and
K.W. Han†
Chung-Shan Institute of Science and Technology Taiwan, China

Introduction

IN the current literature, most of the methods for model reduction are based upon the assumption that the original system has constant parameters.[1,2] Unfortunately, these methods cannot be applied to systems with parameter variations because the closed-loop system response characteristics may not be acceptable even if the reduced models are stable and accurate for a specific set of constant parameters of the system. Since parameter variations are unavoidable in commonly used physical systems, the objective of this Note is to propose a method for reducing the effects due to parameter variations.

This note extends the authors' method[3] to obtain reduced models that can approximate the original transfer function at $s=0$ and ∞, and at some desirable points on the frequency response curve of the original system. It will be shown subsequently in this Note that by a proper selection of these desirable points, such as the phase-crossover frequency and/or peak point of the frequency response curve, the effects of model reduction on control systems can be reduced with parameter variations.

Received July 30, 1984; revision received Nov. 12, 1984.

*Ph.D. Candidate, Institute of Electronics.
†Senior Scientist.

The Proposed Method

Let the original transfer function and reduced model be

$$G(s)=\frac{A_{21}+A_{22}s+\ldots+A_{2n}s^{n-1}}{A_{11}+A_{12}s+\ldots+A_{1,n+1}s^{n}} \tag{1}$$

and

$$R(s)=(\omega_1,\omega_2,\ldots,\omega_m)R[r-1,r]_j^i(s)$$
$$=\frac{d_0+d_1s+\ldots+d_{r-1}s^{r-1}}{c_0+c_1s+\ldots+c_rs^r} \tag{2}$$

respectively. In Eq. (2), r and $r-1$ represent the numbers of poles and zeros of $R(s)$, respectively; i and j the numbers of terms of the continued-fraction expansion of $G(s)$ about $s=0$ and ∞, respectively; and $\omega_1,\omega_2,\ldots,\omega_m$ the frequencies at which the frequency response of $G(s)$ is matched by that of $R(s)$. The procedure for finding the reduced models is as follows.

Step 1. Expand $G(s)$ about $s=0$ for i (even number) times, i.e.,

$$G(s)=\left[h_1+\left[\frac{h_2}{s}+\left[\ldots+\left[\frac{h_i}{s}+\frac{H_N(s)}{H_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \tag{3}$$

where $H_N(s)$ and $H_D(s)$ are the numerator and denominator of the remainder of the continued fraction, respectively, and defined as

$$H_N(s)=A_{i+2,1}+A_{i+2,2}s+\ldots+A_{i+2,n-i/2}s^{n-1-i/2} \tag{4}$$

and

$$H_D(s)=A_{i+1,1}+A_{i+1,2}s+\ldots+A_{i+1,n+1-i/2}s^{n-i/2} \tag{5}$$

Step 2. Reverse the polynomial sequences in Eqs. (4) and (5), and continue to expand Eq. (3) about $s=\infty$ for j (even number) times; then one has

$$\frac{H_N(s)}{H_D(s)}=\left[E_1s+\left[E_2+\left[\ldots+\left[E_j+\frac{F_N(s)}{F_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \tag{6}$$

where

$$F_N(s)=A_{i+j+2,n-(i+j)/2}s^{n-1-(i+j)/2}+\ldots$$
$$+A_{i+j+2,2}s+A_{i+j+2,1} \tag{7}$$

and

$$F_D(s)=A_{i+j+1,n+1-(i+j)/2}s^{n-(i+j)/2}+\ldots$$
$$+A_{i+j+1,2}s+A_{i+j+1,1} \tag{8}$$

Step 3. Let

$$\frac{T_N(s)}{T_D(s)}=\frac{y_{m-1}s^{m-1}+y_{m-2}s^{m-2}+\ldots+y_1s+y_0}{x_ms^m+x_{m-1}s^{m-1}+\ldots+x_1s+x_0} \tag{9}$$

with

$$y_{m-1}=1 \tag{10}$$

where m is the number of points on the frequency response curve of $G(s)$ to be matched by $R(s)$. Let

$$\left.\frac{T_N(s)}{T_D(s)}\right|_{s=j\omega_k}=\left.\frac{F_N(s)}{F_D(s)}\right|_{s=j\omega_k}=r_k+jm_k\quad k=1,2,\ldots,m \tag{11}$$